GENERALIZED ELLIPTIC FUNCTIONS AND THEIR APPLICATION TO A NONLINEAR EIGENVALUE PROBLEM WITH p-LAPLACIAN

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Dedicated to Professor Yoshio Yamada on occasion of his 60^{th} birthday

Abstract. The Jacobian elliptic functions are generalized and applied to a nonlinear eigenvalue problem with p-Laplacian. The eigenvalue and the corresponding eigenfunction are represented in terms of common parameters, and a complete description of the spectra and a closed form representation of the corresponding eigenfunctions are obtained. As a by-product of the representation, it turns out that a kind of solution is also a solution of another eigenvalue problem with $p/2$-Laplacian.

1. Introduction

In this paper we generalize the Jacobian elliptic functions and apply them to a nonlinear eigenvalue problem

$$(\mathrm{PE}_{pq}) \qquad \begin{cases} (\phi_p(u'))' + \lambda \phi_q(u)(1-|u|^q) = 0, & t \in (0,T), \\ u(0) = u(T) = 0, \end{cases}$$

where $T,\ \lambda > 0,\ p,\ q > 1$ and $\phi_m(s) = |s|^{m-2}s\ (s \neq 0),\ = 0\ (s = 0)$.

Problem (PE_{pq}) appears frequently in various articles as stationary problems. In particular, the equation for $p = q = 2$ is called, e.g., the Allen-Cahn equation, the Chafee-Infante equation [3], and a bistable reaction-diffusion equation with logistic effect. The equation for $p = 2 < q$ is said to be a bistable reaction-diffusion equation with Allee effect. In case $p = n$ and $q = 2$ with an n-dimensional domain, an equation of this type is known as the Euler-Lagrange equation of functional related to models introduced by Ginzburg and Landau for the study of phase transitions (cf. Problem 17 in [2]).

As to (PE_{pq}) for general $p > 1$, we have to mention the work [8] by Guedda and Véron [8]. They showed that if $p = q > 1$ then there exists a positive increasing sequence $\{\lambda_n\}$ such that a pair of solutions $\pm u_n$ of

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(PE_{pq}) with $(n-1)$-zeros $z_j = jT/n$ $(j = 1, 2, \ldots, n-1)$ bifurcates from the trivial solution at $\lambda = \lambda_n$ and $|u_n| \to 1$ uniformly on any compact set of $(0, T) \setminus \{z_1, z_2, \ldots, z_{n-1}\}$ as $\lambda \to \infty$. Moreover, they proved that if $p = q > 2$ then for each $n \in \mathbb{N}$ there exists $\Lambda_n > \lambda_n$ such that $\lambda > \Lambda_n$ implies $|u_n| = 1$ on *flat cores* $[z_{j-1} + \frac{T}{2n}(\frac{\Lambda_n}{\lambda})^{1/p}, z_j - \frac{T}{2n}(\frac{\Lambda_n}{\lambda})^{1/p}]$ $(j = 1, 2, \ldots, n)$ of u_n, where $z_0 = 0$ and $z_n = T$. This is a great contrast to case $1 < p = q \leq 2$, where $|u_n| < 1$ in $[0, T]$. Since the equation in (PE_{pq}) is autonomous, if u_n $(n \geq 2)$ has flat cores, then there exists uncountable solution with $(n-1)$-zeros near u_n, which is produced by expanding and contracting the flat cores with preserving its total length $T(1-(\frac{\Lambda_n}{\lambda})^{1/p})$. In this sense, the n-th branch (λ, u_n) bifurcating from $(\lambda_n, 0)$ causes the second bifurcation at $(\Lambda_n, u_{\Lambda_n})$ for each $n \geq 2$.

The phenomena of flat core in [8] above was generalized to case $p > 2$ and $q > 1$ by the author and Yamada [11]. They also studied change in bifurcation depending on the relation between p and q (as far as the first bifurcation is concerned, their proof can be applied to case $1 < p \leq 2$), and showed that for each $n \in \mathbb{N}$, if $p > q$ then there exists a pair of solutions $\pm u_n$ of (PE_{pq}) with $(n-1)$-zeros for $\lambda > 0$; if $p = q$ then there exists $\lambda_n > 0$ such that (PE_{pq}) has no solution with $(n-1)$-zeros for $\lambda \leq \lambda_n$ and (PE_{pq}) has a pair of solutions $\pm u_n$ for $\lambda > \lambda_n$ (the same result as [8]); if $p < q$ then there exists $\lambda_n^* > 0$ such that (PE_{pq}) has no solution with $(n-1)$-zeros for $\lambda < \lambda_n^*$ and (PE_{pq}) has a pair of solutions $\pm u_n$ for $\lambda = \lambda_n^*$ and (PE_{pq}) has two pairs of solutions $\pm u_n$, $\pm v_n$ satisfying $|u_n(t)| > |v_n(t)|$ with $t \neq z_j$ $(j = 0, 1, \ldots, n)$ for $\lambda > \lambda_n^*$. In this sense, the point $(\lambda_n^*, u_{\lambda_n^*})$ causes the *spontaneous bifurcation*. In any case, each solution u_n has flat cores for sufficiently large λ.

The purpose of this paper is to obtain a complete description of the spectra and a closed form representation of the corresponding eigenfunctions of (PE_{pq}), while the studies [8] and [11] above are done in the way of phase-plane analysis and no exact solution is given there.

For the description and representation, we first recall that the Jacobian elliptic function $\mathrm{sn}(t, k)$ with modulus $k \in [0, 1)$ satisfies

$$u'' + u(1 + k^2 - 2k^2u^2) = 0. \tag{1.1}$$

(e.g., Example 4 of p. 516 in the book [13] of Whittaker and Watson). Eq. (1.1) reminds that the solution of nonlinear eigenvalue problem (PE_{pq}) with $p = q = 2$ can be represented explicitly by using $\mathrm{sn}(t, k)$. Indeed, for any given $k \in (0, 1)$, the set of eigenvalues of (PE_{22}) is given

by

$$\lambda_n(k) = (1+k^2)\left(\frac{2nK(k)}{T}\right)^2 \tag{1.2}$$

for each $n \in \mathbb{N}$, with corresponding eigenfunctions $\pm u_{n,k}$, where

$$u_{n,k}(t) = \sqrt{\frac{2k^2}{1+k^2}}\,\mathrm{sn}\left(\frac{2nK(k)}{T}t, k\right) \tag{1.3}$$

and $K(k)$ is the complete elliptic integral of the first kind

$$K(k) = \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}}$$

(cf. Section 2 in [1]). Conversely, all nontrivial solutions are given by Eqs. (1.2) and (1.3), and in particular, it follows from Eq. (1.3) that all solutions satisfy $|u| < 1$.

In our study on (PE_{pq}), after the fashion of Jacobi's $\mathrm{sn}(t,k)$, we introduce a new transcendental function $\mathrm{sn}_{pq}(t,k)$ with modulus $k \in [0,1)$. This satisfies

$$(\phi_p(u'))' + \frac{q}{p^*}\phi_q(u)(1+k^q-2k^q|u|^q) = 0, \tag{1.4}$$

where $p^* := p/(p-1)$. Using $\mathrm{sn}_{pq}(t,k)$, we can obtain a complete description of the set of eigenvalues and the corresponding eigenfunctions of (PE_{pq}) as Eqs. (1.2) and (1.3) with

$$K_{pq}(k) = \int_0^1 \frac{ds}{\sqrt[p]{(1-s^q)(1-k^qs^q)}}.$$

It is important that $K_{pq}(k)$ converges to $K_{\frac{p}{2},q}(0)$ as $k \to 1-0$ if and only if $p > 2$. Indeed,

$$\lim_{k\to 1-0} K_{pq}(k) = \int_0^1 \frac{ds}{(1-s^q)^{\frac{2}{p}}} = K_{\frac{p}{2},q}(0).$$

Similarly, $\mathrm{sn}_{pq}(t,k)$ converges to $\mathrm{sn}_{\frac{p}{2},q}(t,0)$ as $k \to 1-0$. These convergent properties yield the existence of special solutions, not necessarily $|u| < 1$, and we can really construct the solutions of (PE_{pq}) with flat cores. Moreover, $\mathrm{sn}_{\frac{p}{2},q}(t,0)$ satisfies Eq. (1.4) with $k=0$ and p replaced by $p/2$ as well as Eq. (1.4) with $k=1$. Thus, we obtain the following (curious) property: a kind of solution of (PE_{pq}) is also a solution of the nonlinear eigenvalue problem with $p/2$-Laplacian

$$\begin{cases} (\phi_{\frac{p}{2}}(u'))' + \lambda\phi_q(u) = 0, & t \in (0,T), \\ u(0) = u(T) = 0. \end{cases}$$

This paper is organized as follows. In Section 2, we introduce a generalized trigonometric function $\sin_{pq}(t)$ given by Drábek and Manásevich [7] and define a new transcendental function $\mathrm{sn}_{pq}(t,k)$, which is a generalization of the Jacobian elliptic function $\mathrm{sn}(t,k)$ and an extension of $\sin_{pq}(t)$ as $\mathrm{sn}_{pq}(t,0)=\sin_{pq}(t)$. In Section 3, we apply them to nonlinear eigenvalue problems, particularly to the problem considered in [8] and [11], and obtain complete descriptions of the set of eigenvalues and the corresponding eigenfunctions.

2. Transcendental Functions

2.1. **Generalized trigonometric functions.** Generalized trigonometric functions were introduced by Drábek and Manásevich [7] (see also [6]). For $\sigma \in [0,1]$, we define (in a slightly different way from [7])

$$\operatorname{arcsin}_{pq}(\sigma) := \int_0^\sigma \frac{ds}{(1-s^q)^{\frac{1}{p}}}, \tag{2.1}$$

where $p>1,\ q>0$. Letting $s=z^{1/q}$, we have

$$\operatorname{arcsin}_{pq}(\sigma) = \frac{1}{q}\int_0^{\sigma^q} z^{\frac{1}{q}-1}(1-z)^{-\frac{1}{p}}\,dz = \frac{1}{q}\tilde{B}\left(\frac{1}{q},\frac{1}{p^*},\sigma^q\right),$$

where $\tilde{B}(s,t,u)$ denotes the incomplete beta function

$$\tilde{B}(s,t,u) = \int_0^u z^{s-1}(1-z)^{t-1}\,dz.$$

We define the constant π_{pq} as

$$\pi_{pq} := 2\operatorname{arcsin}_{pq}(1) = \frac{2}{q}B\left(\frac{1}{q},\frac{1}{p^*}\right),$$

where $B(s,t)$ denotes the beta function

$$B(s,t) = \tilde{B}(s,t,1) = \int_0^1 z^{s-1}(1-z)^{t-1}\,dz.$$

We have that $\operatorname{arcsin}_{pq} : [0,1] \to [0,\pi_{pq}/2]$, and is strictly increasing. Let us denote its inverse by $\sin_{pq}$. Then, $\sin_{pq} : [0,\pi_{pq}/2] \to [0,1]$ and is strictly increasing. We extend $\sin_{pq}$ to all $\mathbb{R}$ (and still denote this extension by $\sin_{pq}$) in the following form: for $t \in [\pi_{pq}/2, \pi_{pq}]$, we set $\sin_{pq}(t) := \sin_{pq}(\pi_{pq}-t)$, then for $t \in [-\pi_{pq},0]$, we define $\sin_{pq}(t) := -\sin_{pq}(-t)$, and finally we extend $\sin_{pq}$ to all $\mathbb{R}$ as a $2\pi_{pq}$ periodic function.

When $0<p\le 1$, we also define $\operatorname{arcsin}_{pq}$ as Eq. (2.1) for $\sigma \in [0,1)$. We have that $\operatorname{arcsin}_{pq} : [0,1) \to [0,\infty)$, and is strictly increasing. Let us denote its inverse by $\sin_{pq}$. Then, $\sin_{pq} : [0,\infty) \to [0,1)$ and is

strictly increasing. We extend $\sin_{pq}$ to all $\mathbb{R}$ as $\sin_{pq}(t) := -\sin_{pq}(-t)$ for $t \in (-\infty, 0]$ and still denote this extension by $\sin_{pq}$.

Remark 2.1. We immediately find that $\sin_{22}(t) = \sin(t)$ and $\pi_{22} = \pi$ from the properties of the beta function. Moreover, $\sin_{pp}(t) = \sin_p(t)$ and $\pi_{pp} = \pi_p = \frac{2\pi}{p \sin\frac{\pi}{p}}$, where $\sin_p$ and π_p are the generalized sine function and its half-period, respectively, appearing in [4], [5] and [6].

We define for $t \in [0, \pi_{pq}/2]$ (in case $0 < p \leq 1$, for $t \in [0, \infty)$)

$$\cos_{pq}(t) := (1 - \sin_{pq}^q(t))^{\frac{1}{p}},$$

then we obtain

$$\cos_{pq}^p(t) + \sin_{pq}^q(t) = 1,$$
$$\frac{d}{dt}\sin_{pq}(t) = \cos_{pq}(t).$$

Proposition 2.1. *For $p,\ q > 1$, $\sin_{pq}$ satisfies for all $\mathbb{R}$*

$$(\phi_p(u'))' + \frac{q}{p^*}\phi_q(u) = 0. \tag{2.2}$$

Proof. For $t \in (0, \pi_{pq}/2)$ we have

$$\begin{aligned}(\phi_p(u'))' &= (\phi_p(\cos_{pq}(t)))' \\ &= ((1 - \sin_{pq}^q(t))^{\frac{1}{p^*}})' \\ &= \frac{1}{p^*}(1 - \sin_{pq}^q(t))^{-\frac{1}{p}} \cdot (-q \sin_{pq}^{q-1}(t)) \cdot \cos_{pq}(t) \\ &= -\frac{q}{p^*}\phi_q(u).\end{aligned}$$

By symmetry of $\sin_{pq}$, Eq. (2.2) holds true for $t \neq t_n := n\pi_{pq}/2,\ n \in \mathbb{Z}$. Since $\lim_{t \to t_n}(\phi_p(u'))'$ exists, $\phi_p(u')$ is differentiable also at $t = t_n$ and satisfies Eq. (2.2) for all $\mathbb{R}$ in the classical sense. $\square$

2.2. **Generalized Jacobian elliptic functions.** We shall introduce new transcendental functions, which generalize the Jacobian elliptic functions. For $\sigma \in [0, 1]$ and $k \in [0, 1)$, we define

$$\operatorname{arcsn}_{pq}(\sigma) = \operatorname{arcsn}_{pq}(\sigma, k) := \int_0^\sigma \frac{ds}{\sqrt[p]{(1 - s^q)(1 - k^q s^q)}}, \tag{2.3}$$

where $p > 1,\ q > 0$. We define the constant $K_{pq}(k)$ as

$$K_{pq} = K_{pq}(k) := \operatorname{arcsn}_{pq}(1, k) = \int_0^1 \frac{ds}{\sqrt[p]{(1 - s^q)(1 - k^q s^q)}}$$

We have that $\operatorname{arcsn}_{pq} : [0,1] \to [0, K_{pq}]$, and is strictly increasing. Let us denote its inverse by $\operatorname{sn}_{pq}(\cdot) = \operatorname{sn}_{pq}(\cdot, k)$. Then, $\operatorname{sn}_{pq} : [0, K_{pq}] \to [0,1]$ and is strictly increasing. We extend sn_{pq} to all $\mathbb{R}$ (and still denote this extension by sn_{pq}) in the following form: for $t \in [K_{pq}, 2K_{pq}]$, we set $\operatorname{sn}_{pq}(t) := \operatorname{sn}_{pq}(2K_{pq} - t)$, then for $t \in [-2K_{pq}, 0]$, we define $\operatorname{sn}_{pq}(t) := -\operatorname{sn}_{pq}(-t)$, and finally we extend sn_{pq} to all $\mathbb{R}$ as a $4K_{pq}$ periodic function.

When $0 < p \le 1$, we also define $\operatorname{arcsn}_{pq}$ as Eq. (2.3) for $\sigma \in [0,1)$. We have that $\operatorname{arcsn}_{pq} : [0,1) \to [0,\infty)$, and is strictly increasing. Let us denotes its inverse by $\operatorname{sn}_{pq}(\cdot) = \operatorname{sn}_{pq}(\cdot, k)$. Then, $\operatorname{sn}_{pq} : [0,\infty) \to [0,1)$ and is strictly increasing. We extend sn_{pq} to all $\mathbb{R}$ as $\operatorname{sn}_{pq}(t) := -\operatorname{sn}_{pq}(-t)$ for $t \in (-\infty, 0]$ and still denote this extension by sn_{pq}.

The following proposition is crucial to our study.

Proposition 2.2. *For $p,\ q > 0$, K_{pq} is continuous and strictly increasing in $[0,1)$, $2K_{pq}(0) = \pi_{pq}$ and $\operatorname{sn}_{pq}(t,0) = \sin_{pq}(t)$. Moreover,*

$$\lim_{k \to 1-0} 2K_{pq}(k) = \begin{cases} \pi_{\frac{p}{2},q} & \text{if } p > 2, \\ \infty & \text{if } 0 < p \le 2, \end{cases}$$
$$\lim_{k \to 1-0} \operatorname{sn}_{pq}(t,k) = \sin_{\frac{p}{2},q}(t).$$

Proof. The first half is trivial from the definitions of K_{pq} and sn_{pq}. If $p > 2$, then the monotone convergence theorem of Beppo Levi gives

$$\lim_{k \to 1-0} 2K_{pq}(k) = 2\int_0^1 \frac{ds}{(1-s^q)^{\frac{2}{p}}} = 2 \arcsin_{\frac{p}{2},q}(1) = \pi_{\frac{p}{2},q}.$$

If $0 < p \le 2$, then $2K_{pq}(k)$ diverges to ∞ as $k \to 1-0$ by Fatou's lemma.

The last property is proved as follows. By the symmetry of $\operatorname{sn}_{pq}(\cdot,k)$, we may assume $t > 0$. Suppose $p > 2$ and that there exist $t_0,\ \varepsilon > 0$ and $\{k_j\}$ such that $k_j \to 1$ as $j \to \infty$ and

$$|\sigma_{k_j} - \sin_{\frac{p}{2},q}(t_0)| \ge \varepsilon, \tag{2.4}$$

where $\sigma_{k_j} = \operatorname{sn}_{pq}(t_0, k_j)$. Let $n \in \mathbb{Z}$ be the number satisfying $t_0 \in I_n := [n\pi_{\frac{p}{2},q}/2, (n+1)\pi_{\frac{p}{2},q}/2)$ and $j \in \mathbb{N}$ a large number satisfying $t_0 \in I_n(k_j) := [nK_{pq}(k_j), (n+1)K_{pq}(k_j))$. We write $\operatorname{sn}_{pq}^{(n)}(\cdot, k_j)$ as $\operatorname{sn}_{pq}(\cdot, k_j)$ on $I_n(k_j)$ and $\sin_{pq}^{(n)}(\cdot)$ as $\sin_{pq}(\cdot)$ on I_n. Now, since σ_{k_j} is bounded, we can choose a subsequence $\{k_{j'}\}$ of $\{k_j\}$ such that $\sigma_{k_{j'}} \to \sigma$ for some $\sigma \in [-1,1]$ as $j' \to \infty$. Thus, as $j' \to \infty$

$$t_0 = nK_{pq}(k_{j'}) + \operatorname{arcsn}_{pq}(\sigma_{k_{j'}}) \to \frac{n\pi_{\frac{p}{2},q}}{2} + \arcsin_{\frac{p}{2},q}(\sigma),$$

and hence $\sigma = \sin^{(n)}_{\frac{p}{2},q}(t_0)$, which contradicts (2.4). The proof to case $0 < p \leq 2$ is similar and we omit it. $\square$

Remark 2.2. In case $p > 2$, $2K_{pq}(k)$ and $\mathrm{sn}_{pq}(\cdot, k)$ converge to the finite value $\pi_{\frac{p}{2},q}$ and to the finite-periodic function $\sin_{\frac{p}{2},q}$ as $k \to 1-0$, respectively. This is quite different from case $p = 2$, where $2K_{2q}(k)$ diverges to ∞ and $\mathrm{sn}_{22}(t,k)$ converges to the monotone increasing function $\sin_{12}(t) = \tanh(t)$ as $k \to 1-0$.

We define for $t \in [0, K_{pq}]$ (in case $0 < p \leq 1$, for $t \in [0,\infty)$)

$$\mathrm{cn}_{pq}(t) := (1 - \mathrm{sn}^q_{pq}(t))^{\frac{1}{p}},$$
$$\mathrm{dn}_{pq}(t) := (1 - k^q\,\mathrm{sn}^q_{pq}(t))^{\frac{1}{p}},$$

then we obtain

$$\mathrm{cn}^p_{pq}(t) + \mathrm{sn}^q_{pq}(t) = 1,$$
$$\frac{d}{dt}\mathrm{sn}_{pq}(t) = \mathrm{cn}_{pq}(t)\,\mathrm{dn}_{pq}(t).$$

Proposition 2.3. *For $p,\ q > 1$, sn_{pq} satisfies for all $\mathbb{R}$*

$$(\phi_p(u'))' + \frac{q}{p^*}\phi_q(u)(1 + k^q - 2k^q|u|^q) = 0, \tag{2.5}$$

which includes Eq. (2.2) as case $k = 0$.

Proof. For $t \in (0, K_{pq}(k))$ we have

$$\begin{aligned}
(\phi_p(u'))' &= (\phi_p(\mathrm{cn}_{pq}(t)\,\mathrm{dn}_{pq}(t)))' \\
&= (((1 - \sin^q_{pq}(t))(1 - k^q \sin^q_{pq}(t)))^{\frac{1}{p^*}})' \\
&= \frac{1}{p^*}((1 - \sin^q_{pq}(t))(1 - k^q \sin^q_{pq}(t)))^{-\frac{1}{p}} \\
&\qquad \times (-q\sin^{q-1}_{pq}(t) \cdot (1 + k^q - 2k^q \sin^q_{pq}(t))) \cdot \mathrm{cn}_{pq}(t)\,\mathrm{dn}_{pq}(t) \\
&= -\frac{q}{p^*}\phi_q(u)(1 + k^q - 2k^q u^q).
\end{aligned}$$

By symmetry of sn_{pq}, Eq. (2.5) holds true for $t \neq t_n := nK_{pq}(k),\ n \in \mathbb{Z}$. Since $\lim_{t\to t_n}(\phi_p(u'))'$ exists, $\phi_p(u')$ is differentiable also at $t = t_n$ and satisfies Eq. (2.5) for all $\mathbb{R}$ in the classical sense. $\square$

Remark 2.3. Letting $s = \sin_{pq}(t)$ in Eq. (2.3), we have

$$\mathrm{arcsn}_{pq}(\sigma, k) = \int_0^{\mathrm{arcsin}_{pq}(\sigma)} \frac{dt}{\sqrt[p]{1 - k^q \sin^q_{pq}(t)}}.$$

We define the amplitude function $\mathrm{am}_{pq}(\cdot, k) : [0, K_{pq}(k)] \to [0, \pi_{pq}/2]$ by

$$t = \int_0^{\mathrm{am}_{pq}(t,k)} \frac{d\theta}{\sqrt[p]{1 - k^q \sin_{pq}^q(\theta)}},$$

thus sn_{pq} is represented by $\sin_{pq}$ as

$$\mathrm{sn}_{pq}(t, k) = \sin_{pq}(\mathrm{am}_{pq}(t, k)).$$

3. Applications

3.1. **The (p, q)-eigenvalue problem.** Let $T,\ \lambda > 0$ and $p,\ q > 1$. We consider the nonlinear eigenvalue problem

$$(\mathrm{E}_{pq}) \qquad \begin{cases} (\phi_p(u'))' + \lambda\phi_q(u) = 0, & t \in (0, T), \\ u(0) = u(T) = 0. \end{cases}$$

Problem (E_{pq}) has been studied by many authors. In particular, in paper [10] of Ôtani, the existence of infinitely many multi-node solutions was proved by using subdifferential operators method and phase-plane analysis combined with symmetry properties of the solutions. After that, Drábek and Manásevich [7] provided explicit forms of the whole spectrum and the corresponding eigenfunctions for (E_{pq}) (see also [6]). We follow [7] to understand completely the set of all solutions of (E_{pq}).

It will be convenient to find first the solution to the initial value problem

$$\begin{cases} (\phi_p(u'))' + \lambda\phi_q(u) = 0, \\ u(0) = 0,\ u'(0) = \alpha, \end{cases} \tag{3.1}$$

where without loss of generality we may assume $\alpha > 0$.

Let u be a solution to Eq. (3.1) and let $t(\alpha)$ be the first zero point of $u'(t)$. On interval $(0, t(\alpha))$, u satisfies $u(t) > 0$ and $u'(t) > 0$, and thus

$$\frac{u'(t)^p}{p^*} + \lambda\frac{u(t)^q}{q} = \lambda\frac{R^q}{q} = \frac{\alpha^p}{p^*},$$

where $R = u(t(\alpha)) > 0$. Solving for u' and integrating, we find

$$\left(\frac{q}{\lambda p^*}\right)^{\frac{1}{p}} \int_0^t \frac{u'(s)}{(R^q - u(s)^q)^{\frac{1}{p}}}\, ds = t,$$

which after a change of variable can be written as

$$t = \left(\frac{q}{\lambda p^*}\right)^{\frac{1}{p}} \frac{1}{R^{\frac{q}{p}-1}} \int_0^{\frac{u(t)}{R}} \frac{ds}{(1 - s^q)^{\frac{1}{p}}} = \left(\frac{q}{\lambda p^*}\right)^{\frac{1}{p}} \frac{1}{R^{\frac{q}{p}-1}} \operatorname{arcsin}_{pq}\left(\frac{u(t)}{R}\right).$$

Thus we obtain the solution to Eq. (3.1) can be written as

$$u(t) = R\sin_{pq}\left(\left(\frac{\lambda p^*}{q}\right)^{\frac{1}{p}} R^{\frac{q}{p}-1}t\right), \tag{3.2}$$

where $R = (\frac{q}{\lambda p^*})^{\frac{1}{q}}\alpha^{\frac{p}{q}}$.

Theorem 3.1. *All nontrivial solutions of* (E_{pq}) *are given as follows. For any given* $R > 0$*, the set of eigenvalues of* (E_{pq}) *is given by*

$$\lambda_n(R) = \frac{q}{p^*}\left(\frac{n\pi_{pq}}{T}\right)^p R^{p-q} \tag{3.3}$$

for each $n \in \mathbb{N}$*, with corresponding eigenfunctions* $\pm u_{n,R}$*, where*

$$u_{n,R}(t) = R\sin_{pq}\left(\frac{n\pi_{pq}}{T}t\right). \tag{3.4}$$

Proof. For given $R > 0$, by imposing that u in Eq. (3.2) satisfies the boundary conditions in (E_{pq}), we obtain that λ is an eigenvalue of (E_{pq}) if and only if

$$\left(\frac{\lambda p^*}{q}\right)^{\frac{1}{p}} R^{\frac{q}{p}-1}T = n\pi_{pq}, \quad n \in \mathbb{N},$$

and hence Eq. (3.3) follows. Expression (3.4) for the eigenfunctions follows directly from Eq. (3.2). □

3.2. **A perturbed (p,q)-eigenvalue problem.** Let $T,\ \lambda > 0$ and $p,\ q > 1$. We consider the nonlinear eigenvalue problem

$$\begin{cases} (\phi_p(u'))' + \lambda\phi_q(u)(1-|u|^q) = 0, & t \in (0,T), \\ u(0) = u(T) = 0. \end{cases} \tag{PE_{pq}}$$

Problem (PE_{pq}) has been studied by Berger and Fraenkel [1] and Chafee and Infante [3] $(p = q = 2)$, Wang and Kazarinoff [12] and Korman, Li and Ouyang [9] $(p = 2 < q)$, Guedda and Véron [8] $(p = q > 1)$, and Takeuchi and Yamada [11] $(p > 2,\ q > 1)$. However, there is no study providing explicit forms of the whole spectrum and the corresponding eigenfunctions for (PE_{pq}).

As we have done for (E_{pq}), it will be convenient to find first the solution to the initial value problem

$$\begin{cases} (\phi_p(u'))' + \lambda\phi_q(u)(1-|u|^q) = 0, \\ u(0) = 0,\ u'(0) = \alpha, \end{cases} \tag{3.5}$$

where without loss of generality we may assume $\alpha > 0$.

Let u be a solution to Eq. (3.5) and let $t(\alpha)$ be the first zero point of $u'(t)$. On interval $(0, t(\alpha))$, u satisfies $u(t) > 0$ and $u'(t) > 0$, and thus

$$\frac{u'(t)^p}{p^*} + \lambda \frac{F(u)}{q} = \lambda \frac{F(R)}{q} = \frac{\alpha^p}{p^*},$$

where $F(s) = s^q - \frac{1}{2}s^{2q}$ and $R = u(t(\alpha))$. Since we are interested in functions satisfying the boundary condition of (PE_{pq}), it suffices to assume $0 < R \leq 1$, which means $|u| \leq 1$. Moreover, we restrict to $0 < R < 1$ and concentrate solutions satisfying $|u| < 1$ for a while.

Solving for u' and integrating, we find

$$\left(\frac{q}{\lambda p^*}\right)^{\frac{1}{p}} \int_0^t \frac{u'(s)}{\sqrt[p]{F(R) - F(u(s))}}\, ds = t,$$

which after a change of variable can be written as

$$t = \left(\frac{q}{\lambda p^*}\right)^{\frac{1}{p}} \int_0^{\frac{u(t)}{R}} \frac{R}{\sqrt[p]{F(R) - F(Rs)}}\, ds. \tag{3.6}$$

It is easy to verify that

$$F(R) - F(Rs) = F(R)(1 - s^q)\left(1 - \frac{R^q}{2 - R^q} s^q\right),$$

and hence

$$\begin{aligned} t &= \left(\frac{q}{\lambda p^*}\right)^{\frac{1}{p}} \frac{R}{F(R)^{\frac{1}{p}}} \int_0^{\frac{u(t)}{R}} \frac{ds}{\sqrt[p]{(1 - s^q)(1 - k^q s^q)}} \qquad \left(k^q := \frac{R^q}{2 - R^q}\right) \\ &= \left(\frac{q}{\lambda p^*}\right)^{\frac{1}{p}} \frac{R}{F(R)^{\frac{1}{p}}} \operatorname{arcsn}_{pq}\left(\frac{u(t)}{R}, k\right). \end{aligned}$$

Then we obtain that the solution to Eq. (3.5) can be written as

$$u(t) = R \operatorname{sn}_{pq}\left(\left(\frac{\lambda p^*}{q}\right)^{\frac{1}{p}} \frac{F(R)^{\frac{1}{p}}}{R} t, k\right), \tag{3.7}$$

where

$$k = \left(\frac{R^q}{2 - R^q}\right)^{\frac{1}{q}}, \tag{3.8}$$

$$R = \left(\frac{q}{\lambda p^*}\right)^{\frac{1}{q}} \alpha^{\frac{p}{q}} \left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{2q}{\lambda p^*}\alpha^p}\right)^{-\frac{1}{q}}.$$

We first observe the structure of the set of all nontrivial solutions of (PE_{pq}) satisfying $|u| < 1$.

Theorem 3.2 ($|u|<1$)**.** *All nontrivial solutions for $p \in (1,2]$ and all nontrivial solutions with $|u|<1$ for $p>2$ are given as follows. For any given $k \in (0,1)$, the set of eigenvalues of* (PE_{pq}) *is given by*

$$\lambda_n(k) = \frac{q}{p^*}(1+k^q)\left(\frac{2k^q}{1+k^q}\right)^{\frac{p}{q}-1}\left(\frac{2nK_{pq}(k)}{T}\right)^p \tag{3.9}$$

for each $n \in \mathbb{N}$, with corresponding eigenfunctions $\pm u_{n,k}$, where

$$u_{n,k}(t) = \left(\frac{2k^q}{1+k^q}\right)^{\frac{1}{q}} \mathrm{sn}_{pq}\left(\frac{2nK_{pq}(k)}{T}t, k\right). \tag{3.10}$$

Proof. For $k \in (0,1)$ given, we impose that Function (3.7) with $R \in (0,1)$ decided from Eq. (3.8) satisfies the boundary conditions in (PE_{pq}). Then, we obtain that λ is an eigenvalue of (PE_{pq}) if and only if

$$\left(\frac{\lambda p^*}{q}\right)^{\frac{1}{p}} \frac{F(R)^{\frac{1}{p}}}{R} T = 2nK_{pq}(k), \quad n \in \mathbb{N}.$$

From Eq. (3.8) again we have

$$\frac{F(R)^{\frac{1}{p}}}{R} = \left(\frac{2k^q}{1+k^q}\right)^{\frac{1}{p}-\frac{1}{q}} (1+k^q)^{-\frac{1}{p}},$$

and hence we obtain Eq. (3.9). Expression (3.10) for the eigenfunctions follows then directly from Eq. (3.7).

It remains to show that no other nontrivial solution of (PE_{pq}) is obtained when $1 < p \le 2$. Assume the contrary. Then there exist $t_* > 0$ and a nontrivial solution u of (PE_{pq}) with $R = u(t_*) = 1$. However, the right-hand side of Eq. (3.6) with $t = t_*$ diverges because $\sqrt[p]{F(1)-F(s)} = O((1-s^q)^{\frac{2}{p}})$ as $s \to 1-0$. Thus, $t_* = \infty$, which is a contradiction. $\square$

Next we find solutions of (PE_{pq}) with $|u| \le 1$, except the solutions given by Theorem 3.2. From Proposition 2.2, one of solutions of Eq. (3.5) is obtained by $k \to 1-0$ in Eq. (3.7) with Eq. (3.8), namely

$$u(t) = \sin_{\frac{p}{2},q}\left(\left(\frac{\lambda p^*}{2q}\right)^{\frac{1}{p}} t\right).$$

Now we assume $p>2$ and take a number t_* as $(\frac{\lambda p^*}{2q})^{\frac{1}{p}} t_* = \pi_{\frac{p}{2},q}/2$, then u attains 1 at $t = t_*$ (note that it is impossible to obtain such a solution when $1 < p \le 2$). Using this u, we can make the other solutions of Eq. (3.5) as follows. In the phase-plane, the orbit $(u(t), u'(t))$ arrives at the equilibrium point $(1,0)$ at $t = t_*$ and can stay there for any finite time τ before it begins to leave there. Then, the interval $[t_*, t_*+\tau]$ is a

flat core of the solution. Similarly, there is the other equilibrium point $(-1,0)$, where the orbit can stay, and the solution has another flat core of any finite length. Thus we have solutions of Eq. (3.5) attaining ± 1 with any number of flat cores.

Theorem 3.3 ($|u| \le 1$)**.** *Let $p>2$, then all nontrivial solutions are given as follows, in addition to Theorem 3.2. For any given $\tau \in [0,T)$, the set of eigenvalues of* (PE_{pq}) *is given by*

$$\Lambda_n(\tau)=\frac{2q}{p^*}\left(\frac{n\pi_{\frac{p}{2},q}}{T-\tau}\right)^p$$

for each $n \in \mathbb{N}$, with corresponding eigenfunctions $\pm u_{n,\{\tau_i\}}$, where $u_{n,\{\tau_i\}}$ is any function given as follows: for any $\{\tau_i\}_{i=1}^n$ with $\tau_i \ge 0$ and $\sum_{i=1}^n \tau_i = \tau$

(3.11)

$$u_{n,\{\tau_i\}}(t)=\begin{cases}(-1)^{j-1}\sin_{\frac{p}{2},q}\left(\frac{n\pi_{\frac{p}{2},q}}{T-\tau}(t-T_{j-1})\right) & \textit{if}\ \ T_{j-1}\le t\le T_{j-1}+\frac{T-\tau}{2n},\\ (-1)^{j-1} & \textit{if}\ \ T_{j-1}+\frac{T-\tau}{2n}\le t\le T_j-\frac{T-\tau}{2n},\\ (-1)^{j-1}\sin_{\frac{p}{2},q}\left(\frac{n\pi_{\frac{p}{2},q}}{T-\tau}(T_j-t)\right) & \textit{if}\ \ T_j-\frac{T-\tau}{2n}\le t\le T_j,\\ j=1,2,\dots,n,\end{cases}$$

where $T_0=0$ and $T_j=\frac{(T-\tau)j}{n}+\sum_{i=1}^j \tau_i$ for $j=1,2,\dots,n$.

Proof. For each $n\in\mathbb{N}$, it suffices to construct solutions with $(n-1)$-zeros. Let $\tau\in[0,T)$. They are all generated by the eigenvalue and the corresponding eigenfunction of (PE_{pq}) with T replaced by $T-\tau$

$$\Lambda_n(\tau)=\frac{2q}{p^*}\left(\frac{n\pi_{\frac{p}{2},q}}{T-\tau}\right)^p,$$
$$u_{n,\tau}(t)=\sin_{\frac{p}{2},q}\left(\frac{n\pi_{\frac{p}{2},q}}{T-\tau}t\right),$$

which are obtained from Eqs. (3.9) and (3.10) with $k\to 1-0$, respectively. In the phase-plane, the orbit $(u_{n,\tau}(t),u'_{n,\tau}(t))$ goes through the equilibrium points $(\pm 1,0)$ in n-times without staying there as t increases from 0 to $T-\tau$. Therefore, if the orbit stays the i-th equilibrium point for time τ_i, where $\tau_1+\tau_2+\dots+\tau_n=\tau$, then we can obtain Solution (3.11) with n-flat cores in $[0,T]$. $\square$

In Theorems 3.2 and 3.3, we give parameters k and τ to obtain the eigenvalue and the corresponding eigenfunction of (PE_{pq}). Conversely, giving any $\lambda>0$, we can observe the set S_λ of all solutions of (PE_{pq}) by considering the inverses of λ_n and Λ_n.

Theorem 3.4. *Let $p > 1$ and $q > 1$.*

Case $p > q$. For any $\lambda > 0$ there exists a strictly decreasing positive sequence $\{k_j\}_{j=1}^{\infty}$ such that $k_j \to 0$ as $j \to \infty$ and

$$S_\lambda = \{0\} \cup \bigcup_{j=1}^{\infty} \{\pm u_{j,k_j}\}.$$

Case $p = q$. If

$$0 < \lambda \le \frac{q}{p^*}\left(\frac{\pi_{pq}}{T}\right)^p,$$

then $S_\lambda = \{0\}$. If

$$\frac{q}{p^*}\left(\frac{n\pi_{pq}}{T}\right)^p < \lambda \le \frac{q}{p^*}\left(\frac{(n+1)\pi_{pq}}{T}\right)^p, \quad n \in \mathbb{N},$$

then there exists a strictly decreasing positive sequence $\{k_j\}_{j=1}^{n}$ such that

$$S_\lambda = \{0\} \cup \bigcup_{j=1}^{n} \{\pm u_{j,k_j}\}.$$

Case $p < q$. There exists $\lambda_1 > 0$ such that if $0 < \lambda < \lambda_1$, then $S_\lambda = \{0\}$. If $n^p\lambda_1 \le \lambda < (n+1)^p\lambda_1,\ n \in \mathbb{N}$, then there exist a strictly decreasing positive sequence $\{k_j\}_{j=1}^{n}$ and a strictly increasing positive sequence $\{\ell_j\}_{j=1}^{n}$ such that $k_j > \ell_j,\ j = 1, 2, \dots, n-1$ and

$$S_\lambda = \{0\} \cup \bigcup_{j=1}^{n} \{\pm u_{j,k_j}\} \cup \bigcup_{j=1}^{n} \{\pm u_{j,\ell_j}\},$$

where $u_{n,k_n} = u_{n,\ell_n}$ with $k_n = \ell_n$ for $\lambda = n^p\lambda_1$ and $|u_{n,k_n}| > |u_{n,\ell_n}|$ $(t \ne jT/n,\ j = 1, 2, \dots, n-1)$ with $k_n > \ell_n$ otherwise.

In any case, each $k_j,\ \ell_j$ is calculated by Eq. (3.9) for λ_j, and the corresponding solution is given in Form (3.10).

When $1 < p \le 2$, we have $k_j < 1$. When $p > 2$, in addition, if

$$\lambda \ge \frac{2q}{p^*}\left(\frac{m\pi_{\frac{p}{2},q}}{T}\right)^p, \quad m \in \mathbb{N},$$

then for each $j = 1, 2, \dots, m$, the set $\{\pm u_{j,k_j}\}$ above is replaced by $\cup_{\{\tau_i\}}\{\pm u_{j,\{\tau_i\}}\}$, where $\cup_{\{\tau_i\}}$ is the union for all $\{\tau_i\}_{i=1}^{j}$ satisfying $\tau_i \ge 0$ and

$$\sum_{i=1}^{j} \tau_i = T - j\pi_{\frac{p}{2},q}\left(\frac{2q}{\lambda p^*}\right)^{\frac{1}{p}}.$$

The nontrivial solution $u_{j,\{\tau_i\}}$ is given in Form (3.11).

Proof. First we assume $1 < p \leq 2$. In this case, we have already known that all nontrivial solutions of (PE_{pq}) are obtained by Theorem 3.2.

Now we fix $\lambda > 0$. We obtain that λ is the j-th eigenvalue of (PE_{pq}) if and only if from Eq. (3.9) there exists $k \in (0,1)$ such that

$$\frac{T}{2j}\left(\frac{\lambda p^*}{q}\right)^{\frac{1}{p}} = (1+k^q)^{\frac{1}{p}}\left(\frac{2k^q}{1+k^q}\right)^{\frac{1}{q}-\frac{1}{p}} K_{pq}(k) =: \Phi(k). \tag{3.12}$$

Case $p > q$. $\Phi(k)$ is strictly increasing in $(0,1)$ and it follows from Proposition 2.2 that $\Phi(0) = 0$ and $\lim_{k\to 1-0}\Phi(k) = \infty$. Thus, there exists a unique $k = k_j(\lambda)$ satisfying Eq. (3.12). For j and k_j, a unique solution u_{j,k_j} of (PE_{pq}) is obtained by Eq. (3.10).

Case $p = q$. $\Phi(k)$ is strictly increasing in $(0,1)$ and it follows from Proposition 2.2 that $\Phi(0) = \pi_{pq}/2$ and $\lim_{k\to 1-0}\Phi(k) = \infty$. Thus, if

$$\frac{T}{2j}\left(\frac{\lambda p^*}{q}\right)^{\frac{1}{p}} > \frac{\pi_{pq}}{2},$$

namely,

$$\lambda > \frac{q}{p^*}\left(\frac{j\pi_{pq}}{T}\right)^{p},$$

then there exists a unique $k = k_j(\lambda)$ satisfying Eq. (3.12). For j and k_j, a unique solution u_{j,k_j} of (PE_{pq}) is obtained by Eq. (3.10).

Case $p < q$. It is clear that $\lim_{k\to +0}\Phi(k) = \lim_{k\to 1-0}\Phi(k) = \infty$. Changing variable $r = \frac{k^q}{1+k^q}$, we can write Φ as

$$\Psi(r) = \int_0^1 \frac{(1+s^q)^{\frac{1}{p}-\frac{1}{q}}}{(1-s^q)^{\frac{1}{p}}}\psi((1+s^q)r)\,ds, \quad r \in (0,1/2),$$

where $\psi(t) = t^{\frac{1}{q}-\frac{1}{p}}(1-t)^{-\frac{1}{p}}$. It is easy to see that ψ is convex in $(0,1)$ because $\psi(t) > 0$ and

$$(\log\psi(t))'' = \left(\frac{1}{p}-\frac{1}{q}\right)\frac{1}{t^2} + \frac{1}{p}\frac{1}{(1-t)^2} > 0.$$

Then, Ψ is twice-differentiable in $(0,1/2)$ and

$$\Psi''(r) = \int_0^1 \frac{(1+s^q)^{\frac{1}{p}-\frac{1}{q}+2}}{(1-s^q)^{\frac{1}{p}}}\psi''((1+s^q)r)\,ds > 0.$$

Thus, Ψ is convex and there exists $k_* \in (0,1)$ such that $\Phi(k_*)$ is the only one critical value, and hence the minimum of Φ in $(0,1)$.

If

$$\frac{T}{2j}\left(\frac{\lambda p^*}{q}\right)^{\frac{1}{p}} = \Phi(k_*),$$

namely,

$$\lambda = j^p\lambda_1 := \frac{q}{p^*}\left(\frac{2j\Phi(k_*)}{T}\right)^p,$$

then k_* satisfies Eq. (3.12). For j and k_*, a unique solution u_{j,k_*} of (PE$_{pq}$) is obtained by Eq. (3.10). Moreover, if

$$\frac{T}{2j}\left(\frac{\lambda p^*}{q}\right)^{\frac{1}{p}} > \Phi(k_*),$$

namely, $\lambda > j^p\lambda_1$, then there exist $k = k_j(\lambda)$ and $\ell_j(\lambda)$ such that

$$k_j(\lambda) = \Phi^{-1}\left(\frac{T}{2j}\left(\frac{\lambda p^*}{q}\right)^{\frac{1}{p}}\right) \in (k_*, 1),$$
$$\ell_j(\lambda) = \Phi^{-1}\left(\frac{T}{2j}\left(\frac{\lambda p^*}{q}\right)^{\frac{1}{p}}\right) \in (0, k_*).$$

For j, k_j and ℓ_j, solutions u_{j,k_j} and u_{j,ℓ_j} of (PE$_{pq}$) are obtained by Eq. (3.10).

Next, we assume $p > 2$. In any case, a similar proof as above with $\lim_{k\to 1-0}\Phi(k) = 2^{\frac{1}{p}-1}\pi_{\frac{p}{2},q}$ instead of $\lim_{k\to 1-0}\Phi(k) = \infty$ gives that it is impossible to find $k_m \in (0,1)$ above satisfying Eq. (3.12), provided

$$\lambda \geq \frac{2q}{p^*}\left(\frac{m\pi_{\frac{p}{2},q}}{T}\right)^p, \quad m \in \mathbb{N}.$$

Then, however, for each $j = 1, 2, \dots, m$, we can take $\tau \in [0, T)$ such that

$$\lambda = \frac{2q}{p^*}\left(\frac{j\pi_{\frac{p}{2},q}}{T-\tau}\right)^p,$$

and Theorem 3.3 yields the solutions $u_{j,\{\tau_i\}}$, where $\{\tau_i\}_{i=1}^j$ is any sequence satisfying that $\tau_i \geq 0$, $\sum_{i=1}^j \tau_i = \tau$. $\square$

It follows directly from Representation (3.11) of Theorem 3.3 that a kind of solution of (PE$_{pq}$) with p-Laplacian is also an eigenfunction of (E$_{\frac{p}{2},q}$) with $p/2$-Laplacian.

Corollary 3.1. *Let $p > 2$. For each $n \in \mathbb{N}$, any solution $u_{n,\{\tau_i\}}$ of* (PE_{pq}) *in Theorem 3.3 satisfies*

$$(\phi_{\frac{p}{2}}(u'))' + \frac{(p-2)q}{p}\left(\frac{n\pi_{\frac{p}{2},q}}{T-\tau}\right)^{\frac{p}{2}}\phi_q(u) = 0$$

in the intervals where $|u_{n,\{\tau_i\}}| < 1$, where $\tau = \sum_{i=1}^{n}\tau_i$. In particular, for each $n \in \mathbb{N}$, the solution $u_{n,\{0\}}$ of (PE_{pq}) *with $\tau = 0$ is an eigenfunction of* $(\mathrm{E}_{\frac{p}{2},q})$*, that is,*

$$\begin{cases} (\phi_{\frac{p}{2}}(u'))' + \frac{(p-2)q}{p}\left(\frac{n\pi_{\frac{p}{2},q}}{T}\right)^{\frac{p}{2}}\phi_q(u) = 0, & t \in (0,T), \\ u(0) = u(T) = 0. \end{cases}$$

Moreover, $u_{n,\{0\}}$ is characterized by $u_{n,R}$ with $R = 1$ in Eq. (3.4) *with p replaced by $p/2$.*

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